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Representation of Canonical Commutation Relations Associated with Casimir Effect

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Abstract

The Casimir effect in the case of a quantum scalar field is considered in view of representation theory of canonical commutation relations (CCR) with infinite degrees of freedom. It is shown that a very singular irreducible representation of the CCR over an inner product space $\mathscr{E}$ is associated with the Casimir effect and it is inequivalent to the Fock representation of the CCR over $\mathscr{E}$.

1 Introduction

In 1948, Casimir [10] theoretically predicted the following phenomenon: *two parallel perfectly conducting plates facing each other in the vacuum, even if there is no net charge on each of them, have an attractive force of the magnitude:*

$$\frac{\hbar c\pi^2}{240}\frac{1}{a^4} \approx 1.3\times 10^{-27}\frac{1}{a^4} \quad \mathrm{N/m^2}$$

per unit surface area with a the distance between the two plates (see Fig.1), where $\hbar$ is the reduced Planck constant and c is the speed of light. Since then, this surprising phenomenon is referred to as the Casimir effect and the attractive force is called the Casimir force. Experimentally the Casimir effect was confirmed by Sparnaay [26] in 1958 and Lamoreaux [23] in 1998.

A standard physical interpretation for the Casimir effect is that it comes from the change of the zero-point energies of the quantum radiation field according to the change of space configurations of the system under consideration. With this interpretation, the method of calculation for the Casimir force takes the following heuristic form: the potential energy V for the Casimir force is given by $V = E - E_0$ with zero-point energies E and E_0 given by

$$E := \frac{1}{2}\sum \hbar\omega_I = \text{the zero-point energy in the configuration with two plates}$$

and

$$E_0 := \frac{1}{2}\sum \hbar\omega_0 = \text{the zero-point energy in the free space configuration (no plates)},$$

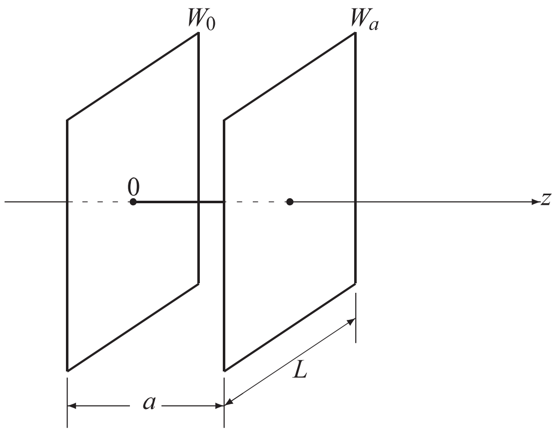

Fig. 1: Casimir effect between two parallel plates W_0 and W_a

where ω_0 (resp. ω_I) is the one-photon energy in the free space configuration (resp. in the configuration with two plates). These quantities are of course divergent and hence meaningless. Hence V is ill-defined in its original form. Therefore one needs a regularization (renormalization) for V to obtain the finite result stated above (for the details of the regularization, see [10] or [22, §3-2-4]).

In 1997, H. Ezawa et al [15] presented a different interpretation from the standard one: the Casimir force is nothing but the Lorentz force acting upon the electric current and charge induced on the plates by the quantum radiation field, suggesting an approach to the Casimir effect without invoking zero-point energy and in terms of representation of canonical commutation relations (CCR) inequivalent to the Fock representation.

A enormous number of physics articles on Casimir effects in various configurations of perfectly conducting bodies has been published, but it seems that there have been few mathematically rigorous studies on Casimir effects, where we mean by "a mathematically rigorous study" that it is a mathematically consistent study using a framework of modern mathematical quantum field theory (e.g., [5, 7, 9, 12, 16, 17, 28]) in which *no use of zero-point energy is made*. Systematic mathematically rigorous studies on the Casimir effect have been given by Herdegen [18, 19] and Herdegen & Stopa [20]. Their approach to Casimir effects is without invoking zero-point energies and uses concepts in algebraic quantum field theory with representations of CCR in regularized forms. A similar point of view was taken in [11].

The present work is motivated by the following philosophy:

> *The Universe uses inequivalent representations of CCR or CAR (canonical anti-commutation relations) to produce characteristic "quantum-macroscopic" phenomena.*

Examples of such phenomena are: (i) Aharonov-Bohm effect ([1] and references therein); (ii) boson masses [3]; (iii) masses of Dirac particles [4]; (iv) Bose-Einstein condensations ([8, 14]

or [2, Chapter 10]); (5) superconductivity [13].

In the case of the Casimir effect, an interaction of macroscopic objects (perfectly conducting plates) with the quantum radiation field is involved. It is known that an effect similar to the Casimir effect may occur also for other quantum fields than the quantum radiation field. The word "Casimir effect" is used for such an effect too. We infer that an inequivalent representation of CCR is associated with Casimir effect. In this paper, we report that this conjecture is right in the case of a quantum scalar field. A new point in our work is that the representation of CCR we construct comes from a *singular* Bogoliubov transformation to which the theory of the standard Bogoliubov transformation (e.g., [27], [21] and references therein) cannot be applied. It is shown that the representation is inequivalent to the Fock representation of the CCR over the same inner product space. In the present paper, we describe only an outline of the results we have obtained. For more details, we refer the reader to paper [6].

Throughout this paper, we use the following notation:

- For a complex inner product space $\mathscr{V}$, we denote by $\langle\ ,\ \rangle_{\mathscr{V}}$ (or simply $\langle\ ,\ \rangle$) the inner product of $\mathscr{V}$ (linear in the second variable and anti-linear in the first) and by $\|\cdot\|_{\mathscr{V}}$ (or $\|\cdot\|$) the norm of $\mathscr{V}$.
- For a linear operator A on a Hilbert space $\mathscr{H}$, we denote by $D(A)$ the domain of A and by A^* the adjoint of A if A is densely defined (i.e., $D(A)$ is dense in $\mathscr{H}$).
- For two linear operators A and B on $\mathscr{H}$, their commutator is defined by $[A,B] := AB - BA$.
- $\mathfrak{B}(\mathscr{H}) :=$ the set of everywhere defined bounded linear operators on $\mathscr{H}$.
- For a linear operator A on $\mathscr{H}$ and a subset $\mathscr{D} \subset D(A)$, $A\mathscr{D} := \{A\Psi | \Psi \in \mathscr{D}\}$.

2 Representations of the CCR over a Complex Inner Product Space

In this section we recall some concepts in the theory of representations of CCR.

Let $\mathscr{F}$ be a complex Hilbert space and $\mathscr{D}$ be a dense subspace of $\mathscr{F}$. Let $\mathscr{V}$ be a complex inner product space.

Definition 2.1 Suppose that, for each $f \in \mathscr{V}$, a densely defined closed linear operator $C(f)$ on $\mathscr{F}$ is given. Then the triple $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^* | f \in \mathscr{V}\})$ is called a representation of the CCR over $\mathscr{V}$ if the following (i)–(iii) hold:

(i) (invariance) For all $f \in \mathscr{V}$, $\mathscr{D} \subset D(C(f)) \cap D(C(f)^*)$. $C(f)\mathscr{D} \subset \mathscr{D}$, $C(f)^*\mathscr{D} \subset \mathscr{F}$,

(ii) (anti-linearity) For all $f, g \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{C}$, $C(\alpha f + \beta g) = \alpha^* C(f) + \beta^* C(g)$ on $\mathscr{D}$, where, for $z \in \mathbb{C}$, z^* denotes the complex conjugate of z.

(iii) (CCR over $\mathscr{V}$) For all $f, g \in \mathscr{V}$,

$$[C(f), C(g)^*] = \langle f, g \rangle_{\mathscr{V}}, \quad [C(f), C(g)] = 0 \quad \text{on } \mathscr{D}.$$

Definition 2.2 Two representations $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^* | f \in \mathscr{V}\})$ and $(\mathscr{F}', \mathscr{D}', \{C'(f), C'(f)^* \,|\, f \in \mathscr{V}\})$ of the CCR over $\mathscr{V}$ are said to be equivalent if there exists a unitary operator $U : \mathscr{F} \to \mathscr{F}'$ such that, for all $f \in \mathscr{V}$,

$$UC(f)U^{-1} = C'(f) \tag{2.1}$$

Remark 2.3 Operator equality (2.1) implies that $UC(f)^*U^{-1} = C'(f)^*$.

Definition 2.4 Let $\mathfrak{A}$ be a set of (not necessarily bounded) linear operators on a Hilbert space $\mathscr{X}$.

(i) The set $\mathfrak{A}$ is said to be reducible if there is a non-trivial closed subspace $\mathscr{M}$ of $\mathscr{X}$ ($\mathscr{M} \neq \{0\}, \mathscr{X}$) such that every $A \in \mathfrak{A}$ is reduced by $\mathscr{M}$ (i.e., $P_{\mathscr{M}}A \subset AP_{\mathscr{M}}$,* where $P_{\mathscr{M}}$ is the orthogonal projection onto $\mathscr{M}$).

(ii) The set $\mathfrak{A}$ is said to be irreducible if it is not reducible.

Definition 2.5 A representation $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^* | f \in \mathscr{V}\})$ of the CCR over $\mathscr{V}$ is said to be reducible (resp. irreducible) if the set $\{C(f), C(f)^* | f \in \mathscr{V}\}$ is reducible (resp. irreducible).

Definition 2.6 For a set $\mathfrak{A}$ of linear operators on a Hilbert space $\mathscr{X}$,

$$\mathfrak{A}' := \{T \in \mathfrak{B}(\mathscr{X}) | TA \subset AT, \forall A \in \mathfrak{A}\}$$

is called the strong commutant of $\mathfrak{A}$.

The following fact is well known [5, Proposition 5.9]:

Lemma 2.7 *Let $\mathfrak{A}$ be a set of linear operators on $\mathscr{X}$.*

(i) *If $\mathfrak{A}' = \mathbb{C}I := \{\alpha I | \alpha \in \mathbb{C}\}$ (I denotes identity), then $\mathfrak{A}$ is irreducible.*

(ii) *If $\mathfrak{A}$ is an irreducible set of densely defined linear operators on $\mathscr{X}$ and $*$-invariant (i.e., $A \in \mathfrak{A} \Longrightarrow A^* \in \mathfrak{A}$), then $\mathfrak{A}' = \mathbb{C}I$.*

3 Representations of CCR in Boson Fock Space

3.1 Boson Fock space

Let $\mathscr{H}$ be a complex Hilbert space and $\otimes_{\mathrm{s}}^n \mathscr{H}$ be the n-fold symmetric tensor product Hilbert space of $\mathscr{H}$ with convention $\otimes_{\mathrm{s}}^0 \mathscr{H} := \mathbb{C}$. Then the boson Fock space over $\mathscr{H}$ is defined as the infinite direct sum Hilbert space of $\otimes_{\mathrm{s}}^n \mathscr{H}$, $n \geq 0$:

$$\begin{aligned}\mathscr{F}_{\mathrm{b}}(\mathscr{H}) &:= \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathscr{H} \\ &= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_{\mathrm{s}}^n \mathscr{H}, n \geq 0, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}.\end{aligned}$$

*For linear operators A and B on $\mathscr{X}$, " $A \subset B$ " means that B is an extension of A, i.e., $D(A) \subset D(B)$ and $A\Psi = B\Psi, \forall \Psi \in D(A)$.

The subspace

$$\mathscr{F}_0(\mathscr{H}) := \{\Psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) | \ \exists n_0 \in \mathbb{N} \text{ such that, for all } n \geq n_0, \ \Psi^{(n)} = 0\}$$

is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ and is called the finite particle subspace of $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$.

For each $f \in \mathscr{H}$, there exists a unique densely defined closed linear operator $A(f)$, called the annihilation operator with test vector $f \in \mathscr{H}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, such that its adjoint $A(f)^*$, called the creation operator with test vector $f \in \mathscr{H}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, is given as follows:

$$\begin{aligned}
&D(A(f)^*) = \left\{ \Psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) | \ \sum_{n=1}^{\infty} n \| S_n(f \otimes \Psi^{(n-1)}) \|^2 < \infty \right\}, \\
&(A(f)^* \Psi)^{(0)} = 0, \\
&(A(f)^* \Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1, \Psi \in D(A(f)^*).
\end{aligned}$$

Basic properties of $A(f)$ and $A(f)^*$ are summarized in the next proposition:

Proposition 3.1

(i) *For all $f \in \mathscr{H}$, $\mathscr{F}_0(\mathscr{H}) \subset D(A(f)) \cap D(A(f)^*)$ and $A(f)$ and $A(f)^*$ leave $\mathscr{F}_0(\mathscr{H})$ invariant.*

(ii) *$\{A(f), A(f)^* | f \in \mathscr{H}\}$ satisfies the CCR over $\mathscr{H}$:*

$$[A(f), A(g)^*] = \langle f, g \rangle_{\mathscr{H}}, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0 \quad (f, g \in \mathscr{H})$$

on $\mathscr{F}_0(\mathscr{H})$.

The vector

$$\Omega_{\mathscr{H}} := \{1, 0, 0, \ldots\} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})$$

is called the Fock vacuum in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. It is easy to see that

$$A(f)\Omega_{\mathscr{H}} = 0, \quad f \in \mathscr{H}.$$

Let T be a non-negative self-adjoint operator on $\mathscr{H}$. Then

$$T^{(n)} := \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \overbrace{T}^{j\text{th}} \otimes I \cdots \otimes I$$

is a non-negative self-adjoint operator on $\otimes_{\mathrm{s}}^n \mathscr{H}$. We set $T^{(0)} := 0$ acting on $\mathbb{C}$. The second quantization operator of T is defined by

$$d\Gamma(T) := \oplus_{n=0}^{\infty} T^{(n)}$$

acting in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. The following theorem is well known:

Theorem 3.2 *The operator $d\Gamma(T)$ is a non-negative self-adjoint operator and, for all $t \in \mathbb{R}$,*

$$e^{itd\Gamma(T)} A(f) e^{-itd\Gamma(T)} = A(e^{itT} f), \quad f \in \mathscr{H}.$$

Remark 3.3 The notion of second quantization operator can be extended to the case where T is a densely defined closable operator on $\mathscr{H}$.

For more details of the theory of boson Fock space, see [5, Chapter 5].

3.2 Fock representation of CCR and related facts

Let $\mathscr{V}$ be a dense subspace of $\mathscr{H}$. Proposition 3.1 implies the following fact:

Proposition 3.4 *The triple*

$$\pi_{\mathrm{F}}(\mathscr{V}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{A(f), A(f)^* | f \in \mathscr{V}\})$$

is an irreducible representation of the CCR over $\mathscr{V}$.

Proof. For a proof of the irreducibility, see [5, Theorem 5.14]. ∎

The representation $\pi_{\mathrm{F}}(\mathscr{V})$ is called the Fock representation of the CCR over $\mathscr{V}$.

Definition 3.5 We say that a densely defined closable linear operator A on a Hilbert space is Hilbert-Schmidt if it is closable and the closure $\overline{A}$ is Hilbert-Schmidt.

The next proposition plays a basic role in the theory of representations of CCR in boson Fock spaces:

Proposition 3.6 *Assume that $\mathscr{H}$ is separable. Let S and T be (not necessarily bounded) linear operators on $\mathscr{H}$ such that there exists a dense subspace $\mathscr{V}$ satisfying*

(i) $\mathscr{V} \subset D(S) \cap D(T)$

(ii) $T_{\mathscr{V}} := T \restriction \mathscr{V}$ *(the restriction of T to $\mathscr{V}$) is injective and* $\mathrm{Ran}\, T_{\mathscr{V}}$ *is dense in* $\mathscr{H}$.

Let C be a conjugation on $\mathscr{H}$ (i.e., C is an anti-linear mapping on $\mathscr{H}$ such that $C^2 = I$ and $\|Cf\| = \|f\|$, $f \in \mathscr{H}$) and suppose that there exists a non-zero vector $\Omega \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that, for all $\Psi \in \mathscr{F}_0(\mathscr{H})$ and $f \in \mathscr{V}$,

$$\langle A(Tf)^*\Psi, \Omega \rangle = -\langle A(CSf)\Psi, \Omega \rangle .$$

Then $ST_{\mathscr{V}}^{-1}$ is Hilbert-Schmidt.

Proof. See [3]. ∎

Remark 3.7 In the case where T and S are in $\mathfrak{B}(\mathscr{H})$, this proposition is essentially known in the context of the theory of standard Bogoliubov transformations (see, e.g., [27], [21] and references therein).

Lemma 3.8 *Let X and Y be (not necessarily bounded) linear operators on $\mathscr{H}$ such that there exists a dense subspace $\mathscr{V} \subset D(X) \cap D(Y)$ and the following equation holds:*

$$\langle Xf, Xg \rangle - \langle Yf, Yg \rangle = \langle f, g \rangle , \quad f, g \in \mathscr{V}. \tag{3.1}$$

Let $X_{\mathscr{V}} := X \restriction \mathscr{V}$. Then $X_{\mathscr{V}}$ is injective and $X_{\mathscr{V}}^{-1}$ is bounded with $\|X_{\mathscr{V}}^{-1}\| \leq 1$.

Proof. It follows from (3.1) that, for all $f \in \mathscr{V}$,

$$\|X_{\mathscr{V}} f\|^2 = \|f\|^2 + \|Yf\|^2 \geq \|f\|^2,$$

which implies the desired result. ∎

The following proposition plays a crucial role in the present work.

Proposition 3.9 *Assume that $\mathscr{H}$ is separable. Let X and Y be as in Lemma 3.8 and suppose that X is unbounded and $X\mathscr{V}$ is dense in $\mathscr{H}$. Then there is no non-zero vector $\Omega \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that*

$$\langle A(Xf)^* \Psi, \Omega \rangle = -\langle A(CYf)\Psi, \Omega \rangle, \quad \Psi \in \mathscr{F}_0(\mathscr{H}), f \in \mathscr{V}.$$

Proof. This proposition can be proved by reductio ad absurdum with applications of Proposition 3.6, Lemma 3.8 and arguments on spectral properties. For the details, see [6, Proposition 3.4]. ∎

3.3 A representation of CCR defined by a singular Bogoliubov transformation

Let T and S be densely defined (not necessarily bounded) linear operators on $\mathscr{H}$ such that there exists a dense subspace $\mathscr{V} \subset D(T) \cap D(S)$ and the following equations hold:

$$\langle Tf, Tg \rangle - \langle Sf, Sg \rangle = \langle f, g \rangle, \tag{3.2}$$

$$\langle Tf, CSg \rangle = \langle Sf, CTg \rangle, \quad f, g \in \mathscr{V}, \tag{3.3}$$

where C is a conjugation on $\mathscr{H}$. Then it is easy to see that, for each $f \in \mathscr{V}$, $A(Tf) + A(CSf)^*$ is closable. Hence one can define a densely defined closed operator $B(f)$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ by

$$B(f) := \overline{A(Tf) + A(CSf)^*}.$$

Equations (3.2) and (3.3) imply that $B(\cdot)$ and $B(\cdot)^*$ satisfy the CCR over $\mathscr{V}$: for all $f, g \in \mathscr{V}$,

$$[B(f), B(g)^*] = \langle f, g \rangle, \ [B(f), B(g)] = 0, \ [B(f)^*, B(g)^*] = 0 \quad \text{on } \mathscr{F}_0(\mathscr{H}).$$

Therefore we have the following fact:

Proposition 3.10

$$\pi_B(\mathscr{V}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{B(f), B(f)^* | f \in \mathscr{V}\}$$

is a representation of the CCR over $\mathscr{V}$.

Remark 3.11

(i) The correspondence T_B: $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is a generalization of the standard Bogoliubov transformation in the sense that S or T may be unbounded and T_B is not necessarily invertible.

(ii) Equation (3.2) implies that T is bounded if and only if S is bounded.

Based on Remark 3.11, we say that the Bogoliubov transformation under consideration is singular if T or S is unbounded (then both T and S are unbounded). We emphasize that, for a singular Bogoliubov transformation, the theory of the standard Bogoliubov transformation (e.g., [27], [21] and references therein) cannot be applied.

Theorem 3.12 *Assume that $\mathscr{H}$ is separable. Suppose that T is bounded and S is not Hilbert-Schmidt. Then $\pi_B(\mathscr{V})$ is inequivalent to any direct sum representations of the Fock representation $\pi_{\mathrm{F}}(\mathscr{V})$. In particular, if $\pi_B(\mathscr{V})$ is irreducible, then $\pi_B(\mathscr{V})$ is inequivalent to $\pi_{\mathrm{F}}(\mathscr{V})$.*

Remark 3.13 Theorem 3.12 is only a slight generalization of a well known fact in the theory of the standard Bogoliubov transformations in that $\mathscr{V}$ is not equal to $\mathscr{H}$ and inequivalence is about direct sum representations of the Fock representation $\pi_{\mathrm{F}}(\mathscr{V})$.

Theorem 3.14 *Assume that $\mathscr{H}$ is separable. Suppose that T is unbounded and $T\mathscr{V}$ is dense in $\mathscr{H}$. Then $\pi_B(\mathscr{V})$ is inequivalent to any direct sum representation of the Fock representation $\pi_{\mathrm{F}}(\mathscr{V})$. In particular, if $\pi_B(\mathscr{V})$ is irreducible, then $\pi_B(\mathscr{V})$ is inequivalent to $\pi_{\mathrm{F}}(\mathscr{V})$.*

Proof. This can be proved by reductio ad absurdum. The essence of the proof is to show that, if $\pi_B(\mathscr{V})$ is equivalent to a direct sum representation of the Fock representation $\pi_{\mathrm{F}}(\mathscr{V})$, then one arrives at a contradiction with Proposition 3.9. ∎

4 A Free Quantum Scalar Field on a Finite Box

In this section, we construct a free relativistic quantum scalar field on the d-dimensional finite box

$$\begin{aligned}\Lambda :&= (0,L)^{d-1} \times (0,L_d)\\ &= \{\boldsymbol{x} = (x_1,\dots,x_d)|x_1,\dots,x_{d-1} \in (0,L), x_d \in (0,L_d)\},\end{aligned}$$

with $d \geq 2$, $L > 0$ and $L_d > 0$. We consider the case where the free quantum scalar field obeys the Dirichlet boundary condition on the boundary of Λ. Hence the Hamiltonian (one-particle Hamiltonian) for a scalar boson is given by

$$h := (-\Delta_{\mathrm{D}} + m^2)^{1/2}, \tag{4.1}$$

where Δ_{D} denotes the Dirichlet Laplacian (e.g., [25, p.263]) acting in $L^2(\Lambda)$ and $m \geq 0$ is the mass of one boson. Then the Hamiltonian of the free quantum scalar field to be constructed is defined by

$$H := d\Gamma(h)$$

acting in the boson Fock space $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$ over $L^2(\Lambda)$.

Remark 4.1 It is well known that $-\Delta_{\mathrm{D}}$ is strictly positive. Hence h is strictly positive even in the case $m = 0$. Therefore h is bijective with $h^{-1} \in \mathfrak{B}(L^2(\Lambda))$. It follows from functional calculus that $h^{-1/2} \in \mathfrak{B}(L^2(\Lambda))$.

We denote by $a(f)$ the annihilation operator with test vector $f \in L^2(\Lambda)$ on $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$ and by $L^2_{\mathrm{real}}(\Lambda)$ the real Hilbert space consisting of real elements in $L^2(\Lambda)$. We take as the time-zero fields the following operators:

$$\phi(f) := \frac{1}{\sqrt{2}}(a(h^{-1/2}f)^* + a(h^{-1/2}f), \quad f \in L^2_{\mathrm{real}}(\Lambda),$$
$$\pi(g) := \frac{i}{\sqrt{2}}(a(h^{1/2}g)^* - a(h^{1/2}g)), \quad g \in D(h^{1/2}) \cap L_{\mathrm{real}}(\Lambda).$$

Then the time t-fields with the Hamiltonian H are defined as follows:

$$\phi(t,f) := e^{itH}\phi(f)e^{-itH}, \quad f \in L^2_{\mathrm{real}}(\Lambda),$$
$$\pi(t,g) := e^{itH}\pi(g)e^{-itH}, \quad g \in D(h^{1/2}) \cap L_{\mathrm{real}}(\Lambda), \quad t \in \mathbb{R}.$$

By Theorem 3.2, we obtain the following proposition:

Proposition 4.2 *For all $t \in \mathbb{R}$,*

$$\phi(t,f) = \phi(e^{ith}f), \quad \pi(t,g) = \pi(e^{ith}g), \quad f \in L^2_{\mathrm{real}}(\Lambda),\ D(h^{1/2}) \cap L^2_{\mathrm{real}}(\Lambda).$$

One can show that $(\phi(t,f), \pi(t,g))$ obeys the following functional field equations:

$$\frac{d}{dt}\phi(t,f)\Psi = \pi(t,f)\Psi,$$
$$\frac{d^2}{dt^2}\phi(t,f)\Psi - \phi(t,\Delta_{\mathrm{D}}f)\Psi + m^2\phi(t,f)\Psi = 0$$

for all $\Psi \in \mathscr{F}_0(L^2(\Lambda))$ and $f \in D(h^2) \cap L^2_{\mathrm{real}}(\Lambda)$, where the time derivatives d/dt and d^2/dt^2 are taken in the strong sense. Thus $\phi(t,\cdot)$ is a free quantum scalar field obeying the Klein-Gordon equation on Λ with Dirichlet boundary condition.

5 A Quantum Scalar Field on Λ with a Partition by a Perfectly-Conducting Wall

We next consider the case where a perfectly conducting wall (plate)

$$W_a := \{\boldsymbol{x} \in \Lambda | x_d = a\}$$

perpendicular to the x_d-axis is placed in Λ with $0 < a < L_d$ (see Fig.2). The box Λ is decomposed as

$$\Lambda = \Lambda_1 \cup W_a \cup \Lambda_2,$$

where

$$\Lambda_1 := \{\boldsymbol{x} \in \Lambda | 0 < x_d < a\}, \quad \Lambda_2 := \{\boldsymbol{x} \in \Lambda | a < x_d < L_d\}.$$

Corresponding to this decomposition, $L^2(\Lambda)$ has the orthogonal decomposition

$$L^2(\Lambda) = L^2(\Lambda_1) \oplus L^2(\Lambda_2).$$

We denote by Δ_ℓ the Dirichlet Laplacian for Λ_ℓ, $\ell = 1,2$.

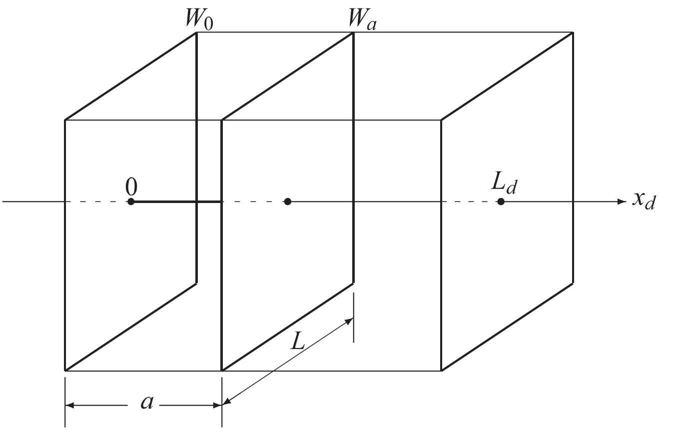

Fig. 2: Box Λ with partition W_a

Remark 5.1 The Dirichlet Laplacian Δ_{D} can not be reduced by $L^2(\Lambda_\ell)$ $(\ell = 1,2)$. This suggests that the presence of W_a may give rise to physics different from that of the system without W_a even if no other interactions exist. This may be a *mathematical origin* of the Casimir effect in the present context.

We take as one-particle Hamiltonian with the wall W_a the following operator:

$$h_a := h_{a,1} \oplus h_{a,2},$$

where

$$h_{a,\ell} := (-\Delta_\ell + m^2)^{1/2}$$

on $L^2(\Lambda_\ell)$.

In what follows, we assume the following:

Assumption (a) The number $L^2/L_d^2 \in \mathbb{Q}$ is a rational number, but a^2/L_d^2 is an irrational one.

5.1 Some properties of the Dirichlet one-particle Hamiltonians and related facts

Let

$$\begin{aligned}\Gamma :&= \left(\frac{\pi}{L}\mathbb{N}\right)^{d-1} \times \frac{\pi}{L_d}\mathbb{N} \\ &= \left\{ \boldsymbol{k} = (k_1, \ldots, k_d) | k_j \in \frac{\pi}{L}\mathbb{N}, j = 1, \ldots, d-1, k_d \in \frac{\pi}{L_d}\mathbb{N} \right\}\end{aligned}$$

For each $\boldsymbol{k} \in \Gamma$ and $\ell = 1, 2$, we define functions $\psi_{\boldsymbol{k}}^{(\ell)}$ on Λ_ℓ as follows:

$$\psi_{\boldsymbol{k}}^{(1)}(\boldsymbol{x}) := \left(\prod_{j=1}^{d-1} \varphi_{k_j}(x_j)\right) \psi_{k_d}^{(1)}(x_d), \quad \boldsymbol{x} \in \Lambda_1,$$
$$\psi_{\boldsymbol{k}}^{(2)}(\boldsymbol{x}) := \left(\prod_{j=1}^{d-1} \varphi_{k_j}(x_j)\right) \psi_{k_d}^{(2)}(x_d), \quad \boldsymbol{x} \in \Lambda_2,$$

where

$$\varphi_{k_j}(x_j) := \sqrt{\frac{2}{L}} \sin(k_j x_j), \quad x_j \in (0, L), \;\; j = 1, \ldots, d-1,$$
$$\psi_{k_d}^{(1)}(x_d) := \sqrt{\frac{2}{a}} \sin \frac{L_d k_d x_d}{a}, \quad x_d \in (0, a),$$
$$\psi_{k_d}^{(2)}(x_d) := \sqrt{\frac{2}{L-a}} \sin \frac{L_d k_d (x_d - a)}{L_d - a}, \quad x_d \in (a, L_d).$$

Then

$$(-\Delta_\ell + m^2)\psi_{\boldsymbol{k}}^{(\ell)} = \omega_\ell(\boldsymbol{k})^2 \psi_{\boldsymbol{k}}^{(\ell)}, \quad \boldsymbol{k} \in \Gamma,$$

where

$$\omega_1(\boldsymbol{k}) := \sqrt{k_1^2 + \cdots + k_{d-1}^2 + (L_d k_d / a)^2 + m^2},$$
$$\omega_2(\boldsymbol{k}) := \sqrt{k_1^2 + \cdots + k_{d-1}^2 + (L_d k_d / (L_d - a))^2 + m^2}.$$

The set $\{\psi_{\boldsymbol{k}}^{(\ell)}\}_{\boldsymbol{k} \in \Gamma}$ is a CONS of $L^2(\Lambda_\ell)$. One has

$$h_{a,\ell} \psi_{\boldsymbol{k}}^{(\ell)} = \omega_\ell(\boldsymbol{k}) \psi_{\boldsymbol{k}}^{(\ell)}$$

Each $f \in L^2(\Lambda)$ is written as

$$f = f^{(1)} + f^{(2)}$$

or $f = (f^{(1)}, f^{(2)})$ with

$$f^{(\ell)} := \chi_{\Lambda_\ell} f \in L^2(\Lambda_\ell),$$

where χ_{Λ_ℓ} is the characteristic function of Λ_ℓ. The direct sum operator of $-\Delta_1$ and $-\Delta_2$

$$-\Delta_{12} := (-\Delta_1) \oplus (-\Delta_2)$$

is a strictly positive self-adjoint operator on $L^2(\Lambda)$. By functional calculus, we have

$$(-\Delta_{12})^{1/2} = (-\Delta_1)^{1/2} \oplus (-\Delta_2)^{1/2}.$$

Lemma 5.2 *$D((-\Delta_{12})^{1/2}) \subset D((-\Delta_{\mathrm{D}})^{1/2})$ and, for all $f \in D((-\Delta_{12})^{1/2})$,*

$$\|(-\Delta_{\mathrm{D}})^{1/2} f\| = \|(-\Delta_{12})^{1/2} f\|. \tag{5.1}$$

In particular, $(-\Delta_{\mathrm{D}})^{1/2}(-\Delta_{12})^{-1/2}$ is an everywhere defined bounded operator on $L^2(\Lambda)$.

Lemma 5.3

(i) $D(h_a) \subset D(h)$ *and* $hh_a^{-1} \in \mathfrak{B}(L^2(\Lambda))$.

(ii) $D(h_a^{1/2}) \subset D(h^{1/2})$ *and* $h^{1/2}h_a^{-1/2} \in \mathfrak{B}(L^2(\Lambda))$.

For convenience, we extend the function $\psi_{\boldsymbol{k}}^{(\ell)}$ to a function on Λ:

$$\tilde{\psi}_{\boldsymbol{k}}^{(\ell)}(\boldsymbol{x}) := \begin{cases} \psi_{\boldsymbol{k}}^{(\ell)}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Lambda_\ell \\ 0 & \text{if } \boldsymbol{x} \in \Lambda \setminus \Lambda_\ell \end{cases}.$$

For all $\alpha \in \mathbb{R}$,

$$h_a^{\alpha} \tilde{\psi}_{\boldsymbol{k}}^{(\ell)} = \omega_\ell(\boldsymbol{k})^{\alpha} \tilde{\psi}_{\boldsymbol{k}}^{(\ell)}, \quad \ell = 1, 2, \boldsymbol{k} \in \Gamma.$$

For each $(\boldsymbol{k}, \boldsymbol{p}) \in \Gamma \times \Gamma$, one has

$$\left\langle \varphi_{\boldsymbol{k}}, \tilde{\psi}_{\boldsymbol{p}}^{(\ell)} \right\rangle = \left(\prod_{j=1}^{d-1} \delta_{k_j p_j} \right) \left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(\ell)} \right\rangle. \tag{5.2}$$

We set

$$c_1 := \frac{L_d}{a}, \quad c_2 := \frac{L_d}{L_d - a}.$$

Lemma 5.4 *For all* $k_d, p_d \in (\pi/L_d)\mathbb{N}$,

$$\left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(1)} \right\rangle = \frac{2}{\sqrt{L_d a}} \frac{(-1)^{L_d p_d} c_1 p_d \sin(a k_d)}{k_d^2 - c_1^2 p_d^2},$$

$$\left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(2)} \right\rangle = -\frac{2}{\sqrt{L_d(L_d - a)}} \frac{c_2 p_d \sin(a k_d)}{k_d^2 - c_2^2 p_d^2}.$$

Lemma 5.5 $D(h_a^{1/2}) \subsetneqq D(h^{1/2})$.

Proof. One can show that, for some $\boldsymbol{k}_0 \in \Gamma$, $\varphi_{\boldsymbol{k}_0} \notin D(h_a^{1/2})$ (this is non-trivial; we use (5.2) and Lemma 5.4 to estimate some quantity). ∎

The following fact shows a singular nature of the pair (h, h_a) of one-particle Hamiltonians:

Lemma 5.6 *The operator* $h^{-1/2}h_a^{1/2}$ *is unbounded.*

Proof. Let $T := h^{-1/2}h_a^{1/2}$. We prove the unboundedness of T by reductio ad absurdum. Suppose that T were bounded. Since $D(T) = D(h_a^{1/2})$, T is densely defined. Hence it is closable and the closure $\overline{T}$ is everywhere defined bounded operator on $L^2(\Lambda)$. We have $T^* = (\overline{T})^*$. Hence $D(T^*) = L^2(\Lambda)$. Since $h^{-1/2}$ is bounded with $D(h^{-1/2}) = L^2(\Lambda)$, it follows that $T^* = (h_a^{1/2})^*(h^{-1/2})^* = h_a^{1/2}h^{-1/2}$. This implies that $D(h^{1/2}) \subset D(h_a^{1/2})$ and $D(T^*) = h^{1/2}D(h_a^{1/2})$. Hence, by Lemma 5.3(ii), $D(h^{1/2}) = D(h_a^{1/2})$. But this contradicts Lemma 5.5. ∎

Lemma 5.7 *Let*

$$S_{\pm} := \frac{1}{2}(h^{-1/2}h_a^{1/2} \pm h^{1/2}h_a^{-1/2}).$$

Then $S_{\pm}$ are unbounded.

Proof. This follows from Lemma 5.3(ii) and Lemma 5.6. ∎

We note that

$$\mathscr{V}_a := D(S_+) = D(S_-) = D(h_a^{1/2}).$$

Lemma 5.8 *For all $f, g \in \mathscr{V}_a$, the following equations hold:*

$$\langle S_+ f, S_+ g\rangle - \langle S_- f, S_- g\rangle = \langle f, g\rangle ,$$
$$\langle S_+ f, S_- g\rangle = \langle S_- f, S_+ g\rangle .$$

Proof. These equations follow from direct computations. ∎

Lemma 5.9 *The range* $\operatorname{Ran} S_+$ *of* S_+ *is dense in* $L^2(\Lambda)$.

5.2 A singular Bogoliubov transformation and a representation of the CCR over a dense subspace

We denote by C_Λ the complex conjugation on $L^2(\Lambda)$:

$$C_\Lambda f := f^*, \quad f \in L^2(\Lambda).$$

We define

$$b(f) := \overline{a(S_+ f) + a(C_\Lambda S_- f)^*}, \quad f \in \mathscr{V}_a. \tag{5.3}$$

It is easy to see that, for all $f, g \in \mathscr{V}_a$,

$$[b(f), b(g)^*] = \langle f, g\rangle ,$$
$$[b(f), b(g)] = 0, \quad [b(f)^*, b(g)^*] = 0$$

on $\mathscr{F}_0(L^2(\Lambda))$. The correspondence : $(a(\cdot), a(\cdot)^*) \mapsto (b(\cdot), b(\cdot)^*)$ is a Bogoliubov transformation. But, by Lemma 5.7, this is a *singular* Bogoliubov transformation.

The linear hull

$$\mathscr{E} := \text{l.h.}\{\tilde{\psi}_{\boldsymbol{k}}^{(\ell)} | \boldsymbol{k} \in \Gamma, \ell = 1, 2\} \tag{5.4}$$

of the subset $\{\tilde{\psi}_{\boldsymbol{k}}^{(\ell)} | \boldsymbol{k} \in \Gamma, \ell = 1, 2\}$ is dense in $L^2(\Lambda)$. For all $\alpha > 0$, $\mathscr{E} \subset D(h_a^\alpha)$ with

$$h_a^\alpha \tilde{\psi}_{\boldsymbol{k}}^{(\ell)} = \omega_\ell(\boldsymbol{k})^\alpha \tilde{\psi}_{\boldsymbol{k}}^{(\ell)}, \quad \boldsymbol{k} \in \Gamma, \ell = 1, 2.$$

Hence

$$h_a^\alpha \mathscr{E} \subset \mathscr{E}. \tag{5.5}$$

Lemma 5.10 *The triple*

$$\pi_b(\mathscr{E}) := (\mathscr{F}_{\mathrm{b}}(L^2(\Lambda)), \mathscr{F}_0(L^2(\Lambda)), \{b(f), b(f)^* | f \in \mathscr{E}\}).$$

is a representation of the CCR over $\mathscr{E}$.

5.3 A quantum scalar field on Λ with W_a

We now introduce the following operators:

$$\phi_b(f) := \frac{1}{\sqrt{2}}(b(h_a^{-1/2}f)^* + b(h_a^{-1/2}f)), \quad f \in L^2_{\mathrm{real}}(\Lambda), \tag{5.6}$$

$$\pi_b(g) := \frac{i}{\sqrt{2}}(b(h_a^{1/2}g)^* - b(h_a^{1/2}g)), \quad g \in D(h_a) \cap L^2_{\mathrm{real}}(\Lambda). \tag{5.7}$$

Lemma 5.11 *For all $f \in D(h^{1/2}) \cap D(h_a) \cap L^2_{\mathrm{real}}(\Lambda)$,*

$$\phi_b(f) = \phi(f), \quad \pi_b(f) = \pi(f) \quad \text{on } \mathscr{F}_0(L^2(\Lambda)) .$$

For each $t \in \mathbb{R}$, we define

$$\phi_b(t,f) := \phi_b(e^{ith_a}f), \quad f \in L^2_{\mathrm{real}}(\Lambda), \tag{5.8}$$

$$\pi_b(t,g) := \pi_b(e^{ith_a}g), \quad g \in D(h_a^{1/2}) \cap L^2_{\mathrm{real}}(\Lambda). \tag{5.9}$$

Note that the time-zero fields coincide:

$$\phi_b(0,f) = \phi(f), \quad \pi_b(0,g) = \pi(g) \quad \text{on } \mathscr{F}_0(L^2(\Lambda)).$$

One can show that the following field equations hold:

$$\frac{d^2}{dt^2}\phi_b(t,f) + \phi_b(t,(-\Delta_{12}+m^2)f) = 0$$

and

$$\frac{d}{dt}\phi_b(t,f) = \pi_b(t,f)$$

on $\mathscr{F}_0(L^2(\Lambda))$,

The quantum fields $\phi_b(t,\cdot)$ and $\pi_b(t,\cdot)$ have the following explicit forms:

$$\phi_b(t,f) = \frac{1}{\sqrt{2}}(a(K(t)f)^* + a(K(t)f)), \quad f \in L^2_{\mathrm{real}}(\Lambda),$$

$$\pi_b(t,g) = \frac{i}{\sqrt{2}}(a(L(t)g)^* - a(L(t)g)), \quad g \in D(h_a) \cap L^2_{\mathrm{real}}(\Lambda)$$

on $\mathscr{F}_0(\mathscr{H}_1)$, where

$$K(t) := h^{-1/2}\cos(th_a) + ih^{1/2}h_a^{-1}\sin(th_a),$$

$$L(t) := h^{1/2}\cos(th_a) + ih^{-1/2}h_a\sin(th_a), \quad t \in \mathbb{R}.$$

One can define operator-valued mappings ϕ_0 and $\phi_{b,0}$ from $\mathbb{R} \times L^2_{\mathrm{real}}(\Lambda)$ to the set of linear operators on $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$ by

$$\phi_0(t,f) := \phi(t,f) \restriction \mathscr{F}_0(L^2(\Lambda)),$$

$$\phi_{b,0}(t,f) := \phi_b(t,f) \restriction \mathscr{F}_0(L^2(\Lambda)), \quad (t,f) \in \mathbb{R} \times L^2_{\mathrm{real}}(\Lambda).$$

The following proposition shows that $\phi_{b,0}$ describes a dynamics different from that of ϕ_0:

Proposition 5.12

$$\phi_0 \neq \phi_{b,0}. \tag{5.10}$$

Remark 5.13 Unfortunately we have been unable to make it clear if there is a self-adjoint operator (a Hamiltonian) H_a on $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$ such that, for all $t \in \mathbb{R}$,

$$e^{itH_a}\phi_b(0,f)e^{-itH_a} = \phi_b(t,f), \quad e^{itH_a}\pi_b(0,f)e^{-itH_a} = \pi_b(t,f)$$

for all f in a suitable dense subspace of $L^2_{\mathrm{real}}(\Lambda)$. We conjecture that such an operator H_a does not exist.

6 Inequivalence of $\pi_b(\mathscr{E})$ to the Fock Representation of the CCR over $\mathscr{E}$ on $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$

The Fock representation of the CCR over $\mathscr{E}$ on $\mathscr{F}_{\mathrm{b}}(L^2(\Lambda))$ is given by

$$\pi_{\mathrm{F}}(\mathscr{E}) := (\mathscr{F}_{\mathrm{b}}(L^2(\Lambda)), \mathscr{F}_0(L^2(\Lambda)), \{a(f), a(f)^* | f \in \mathscr{E}\}).$$

Theorem 6.1 *The representation $\pi_b(\mathscr{E})$ of the CCR over $\mathscr{E}$ is irreducible and inequivalent to the Fock representation $\pi_{\mathrm{F}}(\mathscr{E})$.*

Proof. We first prove the irreducibility of $\pi_b(\mathscr{E})$. Let $T \in \{b(f), b(f)^* | f \in \mathscr{E}\}'$. Then, by (5.6), (5.7) and (5.5), $T \in \{\phi_b(f), \pi_b(f) | f \in \mathscr{E} \cap L^2_{\mathrm{real}}(\Lambda)\}'$. Then we obtain

$$T \in \{\phi(f) \restriction \mathscr{F}_0(L^2(\Lambda)), \pi(f) \restriction \mathscr{F}_0(L^2(\Lambda)) | f \in \mathscr{E} \cap L^2_{\mathrm{real}}(\Lambda)\}',$$

where we have used the fact that $\mathscr{E} \subset D(h_a^{1/2}) \subset D(h^{1/2})$. Recall that $\mathscr{F}_0(L^2(\Lambda))$ is a core for $\phi(f)$ and $\pi(f)$. Hence it follows from a limiting argument that

$$T \in \{\phi(f), \pi(f) | f \in \mathscr{E} \cap L^2_{\mathrm{real}}(\Lambda)\}'.$$

But the right hand side is $\mathbb{C}I$ (essentially due to [24, p.232. Lemma 1]; for a direct proof, see [5, p.289, Example 5.17]). Hence $\{b(f), b(f)^* | f \in \mathscr{E}\}' = \mathbb{C}I$. Thus $\{b(f), b(f)^* | f \in \mathscr{E}\}$ is irreducible.

We next show that $\pi_b(\mathscr{E})$ is inequivalent to $\pi_{\mathrm{F}}(\mathscr{E})$. By Lemmas 5.7–5.9 and the irreducibility of $\pi_b(\mathscr{E})$ proved in the preceding paragraph, we can apply Theorem 3.14 to conclude that $\pi_b(\mathscr{E})$ is inequivalent to $\pi_{\mathrm{F}}(\mathscr{E})$. ∎

7 Concluding Remark

We conjecture that the Casimir force in the present context can be described in terms of the representation $\pi_b(\mathscr{E})$ without invoking zero-point energies.

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